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The Random Parking Problem

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A new approach to the random parking problem is given.

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1. INTRODUCTION

Some statistical problems in physics come naturally in twin versions—one equilibrium version and one irreversible version.

A first example is self-avoiding random walks on a lattice. The designation "walk" suggests a sequentially executed step-by-step process so that the walker each time makes an unbiased choice between neighbor sites not visited before. In this "genuine self-avoiding walk"(1) (also called(2) "kinetic growing walk") two walks of the same total length may have different probabilities of realization. More often, however, the equilibrium version is considered, in which all self-avoiding configurations are assigned a weight that merely depends upon the walk length.

Random configurations of hard spheres constitute a second example.(3) These configurations can be generated by random sequential addition of spheres to the available volume. The twin version to this irreversible procedure is generation of thermal equilibrium configurations. At a given density the two types of configurations are different. The clearest indication of this is that the first process stops at a jamming density less than the density at close packing.

The one-dimensional version of the irreversible hard-sphere problem has been called the *random parking problem.* Cars, all of length l, are

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parked randomly along a road. The resulting jamming density, relative to the maximum density l^{-1}, is given by Rényi's number

$$R=\int_0^\infty dx \exp\left(-2\int_0^x dy\,\frac{1-e^{-y}}{y}\right) \tag{1}$$

which numerically corresponds to a coverage of 74.76%.

The purpose of the present paper is to present a new derivation of R. The final result (1) suggests that the problem is complicated, and Rényi's derivation,[4] which involves a difference-differential equation, is not straightforward. As will be clear below, however, the problem can be solved in a fairly elementary way. As an additional benefit, we obtain the complete time evolution of the density[5] without extra work.

2. FORMULATION

Let the length of a car be l. Then the maximum density is

$$\rho_{\max}=l^{-1}. \tag{2}$$

Adhering to the Mark Kac dictum "Be wise, discretize!," I allow only discretized parking, so that the center of a car can only be positioned at the sites of a regular one-dimensional lattice, with a lattice spacing a given by

$$a=l/(r+1) \qquad (\text{integer } r) \tag{3}$$

Thereby parking is prohibited at any one of the r neighboring sites on either side of a parked car. As a final step, I let the integer $r\to\infty$, thereby reaching the continuum situation.

To specify the problem, assume that the lattice is empty at time $t=0$, and that there is a constant probability $k\,dt$ for an available site to become occupied by a car during the time interval dt. "Available" implies that no car is already parked on the site itself nor on the r nearest-neighbor sites on both sides. Note in passing that for one single isolated site, the probability $p_0(t)$ of finding it unoccupied would decay exponentially,

$$p_0(t)=e^{-kt} \tag{4}$$

At time t we want to determine $\rho(t)$, the average fraction of occupied sites, or, equivalently, the probability that an arbitrarily selected site is occupied. The *coverage* $R(t)$ is the ratio between the corresponding density $\rho(t)/a$ and the maximum density $\rho_{\max}$:

$$R(t)=\rho(t)/a\rho_{\max}=(r+1)\,\rho(t) \tag{5}$$

Finally, Renyi's number (1) is the asymptotic value of the continuum coverage:

$$R = \lim_{t \to \infty} \lim_{r \to \infty} R(t)$$

We must scale down the rate constant k by a factor r,

$$k = c/r \tag{6}$$

in order to have a sensible time evolution in the $r \to \infty$ limit (since there are r parking sites per car length). Here c is an absolute constant independent of r.

3. THE FILLING PROCESS

Consider an arbitrary site i. The probability that it is occupied at time t is denoted by $\rho(t)$, which depends upon time because there is a small probability that the site changes status from unoccupied to occupied during the time interval $(t, t+dt)$. Such a change can only occur if the site itself and the r neighboring sites on both sides are all unoccupied at time t. The probability of this "availability" is

$$P_r(t)\, p_0(t)\, P_r(t) \tag{7}$$

where $P_r(t)$ is the probability that the r terminal units of a *semi-infinite* lattice, initially empty, are all unoccupied at time t. The product form (7) arises from the fact that the three parts of the lattice represented in (7); the site i and the left-hand and the right-hand semi-infinite lattices, are all independent (occupied sites influence neighbors only within the the range r) (see Fig. 1).

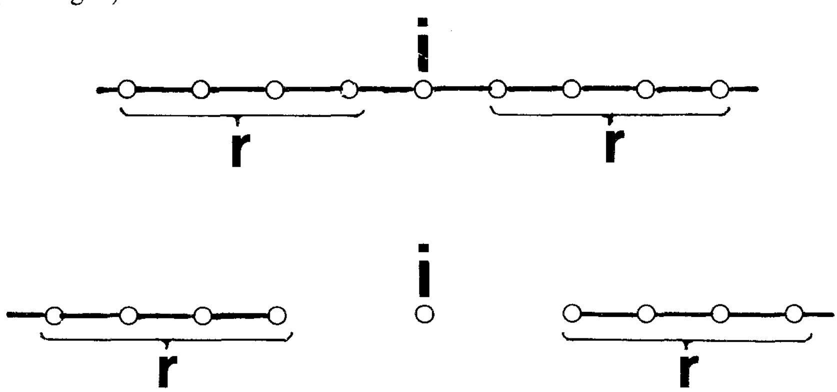

Fig. 1. The three independent parts of configurations with a string of $2r+1$ unoccupied sites.

Thus,

$$\frac{d}{dt}\rho(t)=kP_r(t)\,p_0(t)\,P_r(t) \tag{8}$$

or

$$\rho(t)=k\int_0^t dt' p_0(t')[P_r(t')]^2 \tag{9}$$

It remains to determine $P_n(t)$, the probability that the n terminal sites of a semi-infinite lattice are all unoccupied at time t.

This probability can decrease during the time interval $(t, t+dt)$ because any one of the r terminal sites may become occupied. Note that the ultimate site can be occupied during dt only if the $r+1$ terminal sites are unoccupied at time t, the probability of which is $P_{r+1}(t)$. Similarly the penultimate site can be occupied during dt only if the $r+2$ terminal sites are unoccupied at time t, etc. Thus,

$$\frac{d}{dt}P_r(t)=-k[P_{r+1}(t)+P_{r+2}(t)+\cdots+P_{2r}(t)] \tag{10}$$

We use again independence of the uninfluenced unoccupied sites (beyond the range r), i.e.,

$$P_{r+m}(t)=P_r(t)[p_0(t)]^m \tag{11}$$

Inserting this into (10), we obtain

$$\frac{d}{dt}P_r(t)=-kP_r(t)\sum_{m=1}^{r}[p_0(t)]^m \tag{12}$$

Summation of the geometric series and integration yields

$$P_r(t)=\exp\left\{-\int_0^{kt}\frac{1-[p_0(t')]^r}{1-p_0(t')}\,p_0(t')\,dt'\right\} \tag{13}$$

Insertion of $P_r(t)$ into Eq. (9) yields the average occupation at time t. The coverage (5) equals

$$R(t)=(r+1)k\int_0^t dt'\,p_0(t') \times\exp\left\{-2\int_0^{kt'}\frac{1-[p_0(t'')]^r}{1-p_0(t'')}\,p_0(t'')\,dt''\right\} \tag{14}$$

Introducing the new integration variables

$$y = r[1 - p_0(t'')] = r(1 - e^{-kt''}) \tag{15}$$

and

$$x = r[1 - p_0(t')] = r(1 - e^{-kt'}) \tag{16}$$

as well as $k = c/r$ [Eq. (6)], we find that the coverage (14) takes the form

$$R(t) = \frac{r+1}{r} \int_0^{r[1-\exp(-ct/r)]} dx \times \exp\left\{-2 \int_0^x dy\, \frac{1-(1-y/r)^r}{y}\right\} \tag{17}$$

The continuum limit $r \to \infty$ is now straightforward and yields

$$R(t) = \int_0^{ct} dx \exp\left\{-2 \int_0^x dy\, (1 - e^{-y})/y\right\} \tag{18}$$

which clearly approaches the Rényi number (1) in the $t \to \infty$ limit!

4. CONCLUDING REMARKS

The discrete version (17), which in the simplest case $r = 1$ takes the form

$$R(t) = 1 - \exp[-2(1 - e^{-ct})] \tag{19}$$

may be of interest for discrete systems. Polymer reactions with nearest-neighbor inhibitory effects[6] constitutes an example.

For the problem at hand the existence of a simple solution may merely be of pedagogical value. However, a simplified approach often provides a key to the solution of more complicated problems. For certain complex polymer reaction problems an approach of the present type is essential, but it would lead us too far astray to go into details.

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